B. Second Order Differential Equations:

The second order linear differential equations with constant coefficient has the genral form is:

$$ay'' + by' + cy = F(x) \qquad \ldots(1),$$

where a, b and c are constants.

If $F(x) = 0$ then (1) is called homogenous.

If $F(x) \neq 0$ then (1) is called non homogenous.

Ex:

1) $y''-x^2y'+\sin x\ y=0$ is linear, 2nd order, homo.
2) $y''-(y')^2+ y=\sin x$ is non linear, 2nd order, non homo.
3) $y''+2yy'=\ln x$

a) Homogeneous.

b) Nonhomogeneous.

- ***Undeterminant coefficients.***
- ***Variation of parameters.***

a) The Second order linear homogenous D.Eq. with constant coefficient:

The general form is

$$ay'' + by' + cy = 0 \qquad \ldots(2)$$

where a, b and c are constants.

<u>The general solution</u>

Put $y'=Dy$ and $y''=D^2y$ in eq. (2) (D is an operator)

$\Rightarrow a D^2y+bDy+cy=0$

$\Rightarrow (aD^2 + bD + c)y = 0$ (using D-operator)

now substitute D by r and leave y then

$$ar^2 + br + c = 0$$

is called characteristic equation of the differential equation and the solution of this equation (the roots r) give the solution of the differential equation where

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

There are two values of r :

1- real (equal and not equal).
2- complex.

Case 1: If $b^2 - 4ac \succ 0$ then r_1 and r_2 are distinct ($r_1 \neq r_2$) and real roots, and the general solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case 2*:* If $b^2 - 4ac = 0$ then $r_1 = r_2 = \text{r}$, and the general solution is:

$$y = (c_1 + c_2 x)e^{rx}$$

Case 3: If $b^2 - 4ac \prec 0$ then the roots are two complex conjugate roots $r = \alpha \pm i\beta$, $i = \sqrt{-1}$, and the general solution is:

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

Ex.1: Solve $y'' - 2y' - 3y = 0$

Solution:

$y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0 \quad , \quad \text{y} = 1 \;,\; \text{y}' = r \quad , \quad \text{y}'' = r^2$

$(r+1)(r-3) = 0$

$r + 1 = 0 \quad \Rightarrow \quad r = -1$

$r - 3 = 0 \quad \Rightarrow \quad r = 3$

the general solution is

$y = c_1 e^{-x} + c_2 e^{3x}$

Ex.2: Solve $y'' - 6y' + 9y = 0$

Solution:

$y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = 3$

$\therefore \quad y = (c_1 + c_2 x)e^{3x}$

Ex.3: Solve $y'' + y' + y = 0$

Solution:

$$y'' + y' + y = 0$$

$$r^2 + r + 1 = 0 \qquad a = 1, b = 1, c = 1$$

$$r = \frac{-b \pm \sqrt{1 - 4.1.1}}{2.1}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} \qquad = \frac{-1 \pm \sqrt{3}i}{2}$$

$$r = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i \qquad \alpha = \frac{-1}{2} \quad , \quad \beta = \frac{\sqrt{3}}{2}$$

$$\therefore \quad y = e^{\frac{-1}{2}x}(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x)$$

Exercise: solve

1. $4y''-12y'+5y=0$ ans: $y=c_1e^{(1/2)x}+ c_2e^{(5/2)x}$
2. $3y''-14y'-5y=0$ ans: $y=c_1e^{5x}+ c_2e^{(-1/3)x}$
3. $4y''+y=0$ ans: $y=c_1\cos(x/2)+ c_2\sin(x/2)$
4. $y''-8y'+16y=0$ ans: $y=c_1e^{4x}+ c_2xe^{4x}$
5. $y''+9y=0$ ans: $y=c_1\cos3x+ c_2\sin3x$

b) The Second order linear non homogenous D.Eq. with constant coefficient:

The general form is: $ay'' + by' + cy = F(x)$ …(3)

where a, b and c are constants.

<u>The general solution</u>

If y_h is the solution of the homo. D.Eq. $ay'' + by' + cy = 0$, then the general solution of eq. (3) is:

$y = y_h + y_p$ $\qquad$ y_h (complementary function)

$\qquad\qquad$ y_p (porticular integral)

i) y_h is y homo.
ii) y_p (use the table)

Methods of finding y_p:

There are two methods:

1) Undetermined coefficients:

In this method y_p depends on the roots r_1, and r_2 of characteristic equation and on the form of $F(x)$ in eq. (3) as follows:

$F(x)$	Choice of y_p	M.R.
kx^n nth degree polynomial	$k_n x^n + k_{n-1}x^{n-1} + k_{n-2}x^{n-2} + \cdots + k_0$	0
ke^{px}	ce^{px}	p
$k\sin\beta x$ *or* $k\cos\beta x$	$c_1\cos\beta x + c_2\sin\beta x$	$\mp i\beta$

Note: For repeated term (root), multiply by x.

Ex.1: Use the table to write y_p

1) $F(x) = 3x^2$, $k = 3$, $n = 2$

$y_p = k_2 x^2 + k_1 x + k_0$

2) $F(x)=\frac{-1}{2}e^{-3x}$, $k=\frac{-1}{2}$ $\Rightarrow$ c

$y_p=ce^{-3x}$

3) $F(x)=2\cos 3x$, $k=2$, $\beta=3$

$y_p=c_1\cos 3x+c_2\sin 3x$

4) $F(x)=3x^2-3x+5-2e^{3x}$, $k=-3$, $c=-2$

$y_p=k_2x^2+k_1x+k_0+ce^{3x}$

5) $F(x)=2\cos x-\frac{1}{2}\sin x$

$y_p=c_1\cos x+c_2\sin x$

6) $F(x)=\sin x-\cos 2x$

$y_p=c_1\cos x+c_2\sin x+A\cos 2x+B\sin 2x$

Ex.2: Solve $y''-y'-2y=4x^2$ …. (1)

Solution:

y'' –y'-2y=0

the char. Eq. r^2-r-2=0

$(r+1)(r-2)=0$

$r_1=-1,\ r_2=2$

$y_h=c_1e^{-x}+c_2e^{2x}$

f(x)=$4x^2$ is polynomial of second degree then

$y_p=k_2x^2+k_1x+k_0$ ... (2)

$\Rightarrow\quad y_p'=2k_2x+k_1\quad,\quad y_p''=2k_2$

Substitution gives

$2k_2-(2k_2x+k_1)-2(k_2x^2+k_1x+k_0)=4x^2$

$coeff.of\ x^2:-2k_2=4\quad\Rightarrow\quad k_2=-2$

$coeff.of\ x:-2k_2-2k_1=0\quad\Rightarrow\ k_1=2$

$const:2k_2-k_1-2k_0=0\quad\Rightarrow\ k_0=-3$

$y_p=-2x^2+2x-3$

$y_g=y_h+y_p=(c_1e^{-x}+c_2e^{2x})-2x^2+2x-3$

Ex.3: $y'' - y' - 2y = e^{3x}$

Solution:

$y'' - y' - 2y = e^{3x} \quad .... (1)$

$y'' - y' - 2y = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0 \Rightarrow r_1 = 2, r_2 = -1$

$y_h = (c_1 e^{2x} + c_2 e^{-x})$, Put

$y_p = ce^{3x} \quad .... (2)$

$y'_p = 3ce^{3x} \quad , \quad y''_p = 9ce^{3x}$

Substitute In (1)

$9ce^{3x} - 3ce^{3x} - 2ce^{3x} = e^{3x}$

$9c - 3c - 2c = 1 \Rightarrow 4c = 1 \quad \Rightarrow \quad c = \frac{1}{4}$

In (2)$\Rightarrow \quad y_p = \frac{1}{4} e^{3x}$

$y_g = y_h + y_p = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{4} e^{3x}$

Modification rule قاعدة التعديل

١) اذا كان $F(x) = kx^n$ **وكان احد جذري المعادلة القياسية = ٠ ← يضرب** y_p **السابق في** x.

٢)

a - اذا كان $F(x) = ke^{px}$ **وكان احد جذري المعادلة القياسية = p ← يضرب** y_p **السابق في** x.

b - اذا كان $F(x) = ke^{px}$ **وكان جذري المعادلة القياسية = p ← يضرب** y_p **السابق في** x^2.

٣) اذا كان $F(x) = \begin{cases} k \cos \beta x \\ k \sin \beta x \end{cases}$ **وكان** $\alpha = 0$, $r = \mp i\beta$ **← يضرب** y_p **السابق في** x.

Ex.4: Solve $y'' + y = \sin x$

Solution:

$y'' + y = 0$

$r^2 + 1 = 0$, $r^2 = -1 \Rightarrow r = \pm i$, $\alpha = 0$, $\beta = 1$

$y_h = c_1 \cos x + c_2 \sin x$

$y_p = x(c_3 \cos x + c_4 \sin x)$, $y'_p = x(-c_3 \sin x + c_4 \cos x) + (c_3 \cos x + c_4 \sin x)$

$y''_p = x(-c_3\cos x - c_4\sin x) + (-c_3\sin x + c_4\cos x) + (-c_3\sin x + c_4\cos x)$

Substitution gives

$-2c_3\sin x + 2c_4\cos x = \sin x$

$-2c_3 = 1 \Rightarrow c_3 = -1/2,\ 2c_4 = 0 \Rightarrow c_4 = 0$

$$y_g = c_1\cos x + c_2\sin x - \frac{x}{2}\cos x$$

Exercise: Find the general solution

1) $y'' = 9x^2 + 2x - 1$

2) $y'' - y' - 2y = \sin 2x$

3) $y'' - 5 = 3e^x - 2x + 1$

4) $y'' + 2y' + y = 3e^{-x}$

5) $y'' - y' - 2y = x^2 - x$

2- Variation of parameters.

Let $y_h=c_1u_1+c_2u_2$ be the homogenous solution of $ay''+by'+cy=F(x)$ and the particular solution has the form $y_p=u_1v_1+u_2v_2$ where v_1 and v_2 are unknown functions of x which must be determined, first solve the following linear equations for v'_1 and v'_2:

$v'_1u_1+v'_2u_2=0$

$v'_1u'_1+v'_2u'_2=F(x)$

which can be solved with respect to v'_1 and v'_2 by Grammar rule as follows

$$D=\begin{vmatrix}u_1 & u_2\\ u'_1 & u'_2\end{vmatrix},\quad D_1=\begin{vmatrix}0 & u_2\\ F(x) & u'_2\end{vmatrix},\quad D_2=\begin{vmatrix}u_1 & 0\\ u'_1 & F(x)\end{vmatrix}$$

and $v'_1=\dfrac{D_1}{D},\quad v'_2=\dfrac{D_2}{D}$

by integration of v'_1 and v'_2 with respect to x we can find v_1 and v_2.

Ex.1:

Solve $y''-y'-2y=e^{3x}$ ……. (1)

$y_h=c_1e^{-x}+c_2e^{2x}$, hence

$u_1=e^{-x} \Rightarrow u_1'=-e^{-x}$

$u_2=e^{2x} \Rightarrow u_2'=2e^{2x}$

$y_p=v_1u_1+v_2u_2$

$v_1'u_1+v_2'u_2=0$

$v_1'u_1'+v_2'u_2'=F(x)$

$v_1'(e^{-x})+v_2'(e^{2x})=0$

$v_1'(-e^{-x})+v_2'(2e^{2x})=e^{3x}$

Solving this system by Cramer rule gives

$$D=\begin{vmatrix}e^{-x} & e^{2x}\\ -e^{-x} & 2e^{2x}\end{vmatrix}=3e^{x},\quad D_1=\begin{vmatrix}0 & e^{2x}\\ e^{3x} & 2e^{2x}\end{vmatrix}=-e^{5x},\quad D_2=\begin{vmatrix}e^{-x} & 0\\ -e^{-x} & e^{3x}\end{vmatrix}=e^{2x}$$

$$v'_1=\frac{-e^{5x}}{3e^{x}}=\frac{-1}{3}e^{4x}\Rightarrow v_1=\int\frac{-1}{3}e^{4x}=-\frac{1}{12}e^{4x},$$

$$v'_2=\frac{e^{2x}}{3e^{x}}=\frac{1}{3}e^{x}\Rightarrow v_2=\int\frac{1}{3}e^{x}=\frac{1}{3}e^{x}$$

$$\therefore y_p = -\frac{1}{2}e^{4x}e^{-x} + \frac{1}{3}e^{x}e^{2x} = \frac{1}{4}e^{3x}$$

the general solution is : $y = c_1e^{-x} + c_2e^{2x} + \frac{1}{4}e^{3x}$

Ex.2: solve

$y''+y=\sec x$

Solution:

$y''+y=0$

$r^2+1=0 \Rightarrow r^2=-1 \Rightarrow r = \pm i \qquad \alpha=0,\ \beta=1$

$y_h=c_1\cos x + c_2\sin x,\quad u_1=\cos x,\ u_2=\sin x,\ f(x)=\sec x$

$y_p= v_1u_1+v_2u_2$

$= v_1\cos x + v_2\sin x$ then

$$v_1'(\cos x) + v_2'(\sin x) = 0$$
$$v_1'(-\sin x) + v_2'(\cos x) = \sec x$$

$$D = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1,$$

$$D_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sin x\ \sec x = -\sin x\frac{1}{\cos x} = -\tan x,$$

$$D_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = 1$$

$$v_1' = \frac{-\tan x}{1} = -\tan x \Rightarrow v_1 = \int \frac{-\sin x}{\cos x}dx = \ln|\cos x|$$
$$v'_2 = 1 \Rightarrow v_2 = \int dx = x$$

$y_p = \ln\ |\cos x|\ \cos x + x\sin x$

$y_g = c_1\cos x + c_2\sin x + \ln\ |\cos x|\ \cos x + x\sin x$

Exercise

1. $y''-2y'+y = e^x \ln x$
2. $y''-2y'+y = \frac{e^x}{x^5}$
3. $y''+4y=\sin^2 2x$

Problems: Find the general solution

1- $y'' + 2y' + y = 4e^{-x}\ln x$

2- $y'' + 4y' + 20y = 23\sin t - 15\cos t$ $y(0) = 0$, $y'(0) = -1$

3- $y'' - 4y' + 3y = 4e^{3x}$ $y(0) = -1$, $y'(0) = 3$

4- $y'' + y = x^2 + x$

5- $y'' - 2y' + y = \dfrac{12e^x}{x^3}$

6- $y'' + 4y = 4\sec 2x$

References:

1- calculus & Analytic Geometry (Thomas).

2- Calculus (Haward Anton).

3- Advanced Mathematics for Engineering Studies **(أ. رياض احمد عزت)**

4- Modern Introduction Differential Equations, Schaum's Outline Series.